Technical note

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Revision 1

Stokes waves in AquaSim

aquastructures

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1 Introduction

Stokes waves describe nonlinear, periodic surface gravity waves in inviscid fluids. Unlike linear wave theory, Stokes' perturbation approach accounts for wave steepness, making it suitable for steeper waves encountered in ocean engineering. This tutorial explores the fifth-order Stokes wave theory, focusing on the dispersion relation and kinematic quantities (pressure, velocities, and accelerations) for both finite and infinite water depths.

2 Theorethical background

Stokes wave theory expands the wave solution as a perturbation series in wave steepness, ε = ka, where k is the wave number and a is the amplitude. The fifth-order approximation includes terms up to ε5, providing accurate predictions for moderately steep waves. The theory assumes an inviscid, incompressible fluid with irrotational flow, governed by the Laplace equation and nonlinear boundary conditions at the free surface. For finite depth h, the dispersion relation and kinematics depend on hyperbolic functions (e.g., cosh(kh), tanh(kh)), while in infinite depth (kh → ∞), these simplify to exponential forms. Key quantities include dynamic pressure p, horizontal velocities u and v, vertical velocity w, and accelerations ax, ay, and az

Theoretical Background Water wave theory begins with the simplest model: linear Airy waves, named after George Biddell Airy (Airy 1841) solves assume small wave amplitudes relative to wavelength and depth, leading to a linearized set of governing equations. Under these assumptions, the free surface elevation η(x, t) of a progressive wave in one dimension is sinusoidal:

$$\eta = a\cos(kx - \omega t)$$

where *a* is the wave amplitude, *k* is the wave number (*k = 2π/λ*, with *λ* the wavelength), ω is the angular frequency, and the phase *kx−ωt* determines the wave's propagation. Note that the mathemathical formulation is presented in a fashion where the basic Airy wave elevation follows a cosinus and not sinus. This means that to be consistent with AquaSim the angle ϕ in this presentation is -π/2 relative to the other theory documents. The dispersion relation for Airy waves is:

$$\omega^2 = gk\tanh(kh)$$

where *g* is gravitational acceleration and h is water depth. This relation connects wave frequency to wavelength and depth, with tanh(kh) transitioning from shallow-water *(ω ∝ k)* to deep- water (*ω* ∝ √*k*) behavior as *kh* increases. The linearity assumption simplifies the boundary conditions at the free surface, neglecting higher-order terms in the amplitude a, making Airy waves symmetric and non-steepening. However, real ocean waves often exhibit nonlinearity, especially as wave steepness (*ka*) increases. In 1847, George Gabriel Stokes introduced a perturbation expansion to account for these effects (Stokes1847). Stokes proposed that the surface elevation and velocity potential could be expressed as power series in the wave steepness parameter *ε* = *ka*:

$$\eta = a\cos\phi + \varepsilon a^2 k B_{22}\cos 2\phi + \varepsilon^2 a^3 k^2 B_{33}\cos 3\phi + \cdots$$

$$\Phi = \frac{a\omega}{k}\frac{\cosh k\,(z+h)}{\sinh k\,h}\sin\phi + \text{higher-order terms}$$

where Φ is the velocity potential, $\phi = kx - \omega t + \varphi_0$ is the phase, and B_{nn} are coefficients determined by solving the nonlinear boundary conditions order by order. At first order ($\varepsilon 1$), this reduces to the Airy solution, but higher orders introduce harmonic components (e.g., $\cos 2\phi$, $\cos 3\phi$), causing wave crests to sharpen and troughs to flatten—a hallmark of nonlinear waves. The Stokes expansion assumes an inviscid, incompressible fluid with irrotational flow, gov- erned by the Laplace equation in the fluid domain. The key challenge lies in satisfying the nonlinear kinematic and dynamic boundary conditions at the free surface :

$$Kinematic: \frac{\partial \eta}{\partial t} + \frac{\partial \Phi}{\partial x}\frac{\partial \eta}{\partial x} = \frac{\partial \Phi}{\partial z},$$

$$Dynamic: \frac{\partial \Phi}{\partial t} + \frac{1}{2}|\nabla\Phi|^2 + g\eta =$$

In Airy theory, these are linearized by assuming small η and $\nabla\Phi$, but Stokes retains the nonlinear terms, solving iteratively. At second order, a cos 2ϕ term emerges, adjusting the dispersion relation and kinematics. By fifth order, terms up to ε5 (i.e., cos 5ϕ) are included, providing accuracy for steeper waves (ka ≈ 0.3 or higher). John Fenton advanced this framework in 1985, deriving explicit coefficients and dispersion relations for the fifth-order Stokes wave in both finite and infinite depths Fenton1985. In finite depth, hyperbolic functions like cosh k(z + h) and sinh kh dominate the solutions, reflecting the influence of the seabed. In infinite depth (kh → ∞), these simplify to exponentials (ekz), as the bottom boundary's effect vanishes. Fenton's work provides the D2 and D4 coefficients for the dispersion relation and Bnn coefficients for the surface elevation and kinematics, en- abling practical computations of pressure, velocities, and accelerations—key quantities in ocean engineering and wave-structure interaction studies. This tutorial focuses on the fifth-order Stokes theory, bridging the gap between Airy's linear waves and the nonlinear regime, with equations applicable to both finite and infinite depths

3 Mathematical Formulation

This section outlines the dispertion relation and the expressions for pressure,

3.1 Dispersion Relation

The dispersion relation relates the angular frequency ω to the wave number k. For fifth-order Stokes waves:

3.1.1 Finite Depth

In finite depth, the dispersion relation includes nonlinear corrections:

$$\omega^2 = gk\tanh(kh)\,[1 + (ka)^2 D_2 + (ka)^4 D_4]$$

The dispersion relation relates the angular frequency ω to the wave number k. For fifth-order Stokes waves:

$$D_2 = \frac{9 - 8\cosh^2(kh) + 8\cosh^4(kh)}{8\sinh^4(kh)},\backslash$$

$$D_4 = \frac{384\cosh^6(kh) - 448\cosh^4(kh) + 72\cosh^2(kh) + 9}{128\sinh^8(kh)}.$$

This equation is solved iteratively, as implemented in the bisection method of the subroutine.

3.1.2 Infinite Depth

For deep water *(kh > 10), tanh(kh) → 1*, and the dispersion simplifies to:

$$\omega^2 = gk\left[1 + (ka)^2 + \frac{1}{2}(ka)^4\right].$$

The higher-order terms reflect the nonlinear steepening of deep-water waves.

3.2 Kinematic Quantities

3.2.1 Surface Elevation

The surface elevation $\eta(x, y, t)$ is:

$$\eta = a\cos\phi + a^2kB_{22}\cos 2\phi + a^3k^2B_{33}\cos 3\phi + a^4k^3B_{44}\cos 4\phi + a^5k^4B_{55}\cos 5$$

Where $\phi = \omega t - k(x\cos\beta + y\sin\beta) + \varphi_0$, β is the wave direction, and the coefficients B_{nn} are:

$B_{11} = 1$ $B_{22} = \frac{1}{2}$ $B_{33} = \frac{3}{8}\varepsilon^2$ $B_{44} = \frac{1}{3}\varepsilon^3$ $B_{55} = \frac{5}{16}\varepsilon^4$

3.2.2 Finite Depth

The pressure and velocities at depth $z\ (-h \leq z \leq \eta)$ are

$$p = \rho g a B_{11}\frac{\cosh k\,(z+h)}{\sinh k\,h}\cos\phi + \rho g a^2 B_{22}\frac{\cosh 2\,k(z+h)}{\sinh 2\,kh}\cos 2\phi + \cdots,\backslash$$

$$u = \omega a B_{11}\frac{\cosh k\,(z+h)}{\sinh k\,h}\cos\phi + 2\omega a^2 B_{22}\frac{\cosh 2\,k(z+h)}{\sinh 2\,kh}\cos 2\phi + \cdots,\backslash$$

$$w = -\omega a B_{11}\frac{\sinh k\,(z+h)}{\sinh k\,h}\sin\phi - 2\omega a^2 B_{22}\frac{\sinh 2\,k(z+h)}{\sinh 2\,kh}\sin 2\phi + \cdots,$$

with acc

elerations derived as $a_x = \partial u/\partial t$, $a_y = a_x \sin\beta$, $a_z = \partial w/\partial t$ up to fifth order.

3.2.3 Ininite Depth

The pressure and velocities at depth z $(-h \leq z \leq \eta)$ are

$$p = \rho g a B_{11} e^{kz} \cos\phi + \rho g a^2 B_{22} e^{2kz} \cos 2\phi + \cdots$$

$$u = \omega a B_{11} e^{kz} \cos\phi + 2\omega a^2 B_{22} e^{2kz} \cos 2\phi + \cdots$$

$$w = -\omega a B_{11} e^{kz} \sin\phi - 2\omega a^2 B_{22} e^{2kz} \sin 2\phi + \cdots$$

with accelerations similarly computed up to fifth order

4 Examples

4.1 Case 1 beam at water line

A beam in the water line seen in Figure 1 is used as base case

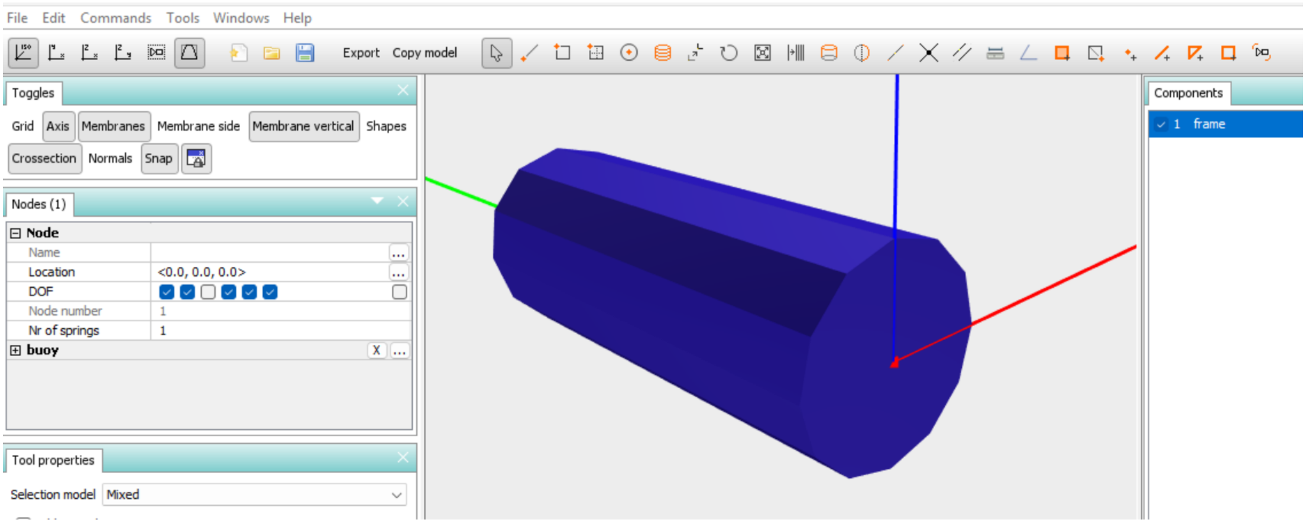


Figure 1 Beam at surface

The environmental data is given in Table 1.

Table 1 Environment data

Wave amplitude, a, [m]	1
Wave period, T, [s]	4
Wave numer, k, [1/m]	0.25

4.1.1 Case 1 version 1, springs

In this case mass and volume of the beam is set to 0 and buoys are attached to the nodes as shown in Figure 2.

Spring	
Name	buoy
Type	Buoy
Maximum force	0.0 N
Stiffness	
X	0.0 N/m
Y	0.0 N/m
Z	1E5 N/m
Stiffness rotation	

Figure 2 Buoys at nodes

Figure 3 shows results. In Figure 3

Airy surface analytic is the surface according to prediction by Airy wave.

Vertical displacemen is the vertical displacement of the beam

Stokes surface analytic is the wave elevation of the Stokes 5th order wave.

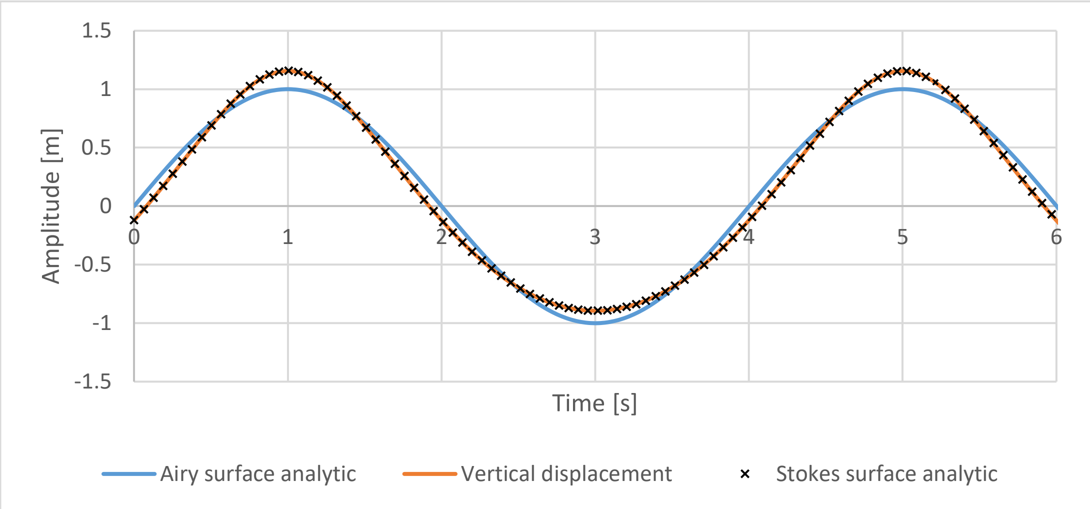
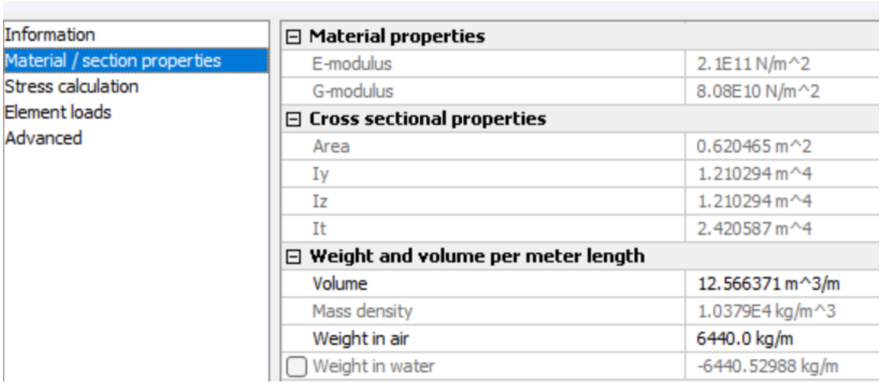


Figure 3

As seen from Figure 3 the beam follows the surface exactly as it is supposed to.

4.1.2 Case 1 version 2

In this case the beam is modelled with volume and weigh as shown in Figure 4.



Material properties	
E-modulus	2.1E11 N/m^2
G-modulus	8.08E10 N/m^2
Cross sectional properties	
Area	0.620465 m^2
Iy	1.210294 m^4
Iz	1.210294 m^4
It	2.420587 m^4
Weight and volume per meter length	
Volume	12.566371 m^3/m
Mass density	1.0379E4 kg/m^3
Weight in air	6440.0 kg/m
Weight in water	-6440.52988 kg/m

Figure 4

In this case the cmparison was done with wave period, T = 5 s. Comparison is shown in Figure 5 and results shown that for the Stokes wave the displacement upwards is larger and downwards less which is logical given the shape and buoyancy of the Stokes wave relative to the Airy wave.

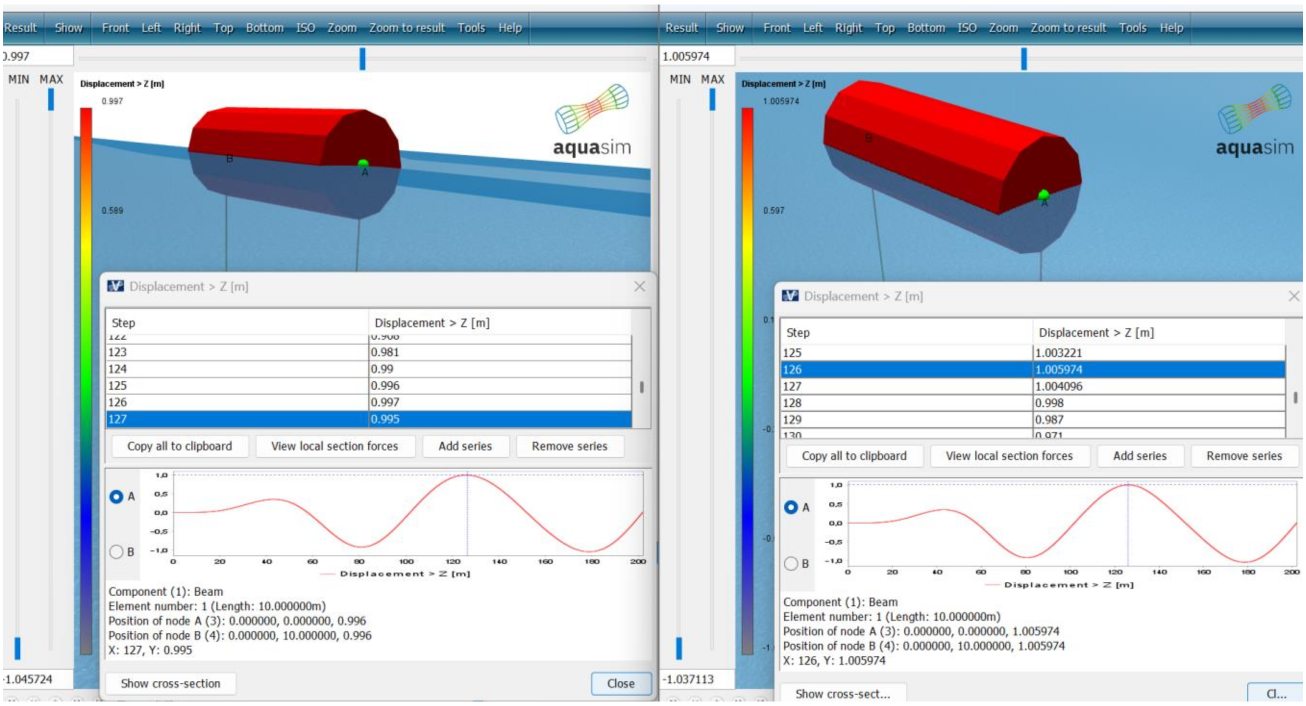


Figure 5 Airy wave to the left and Stokes wave to the right.

5 References

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